

Correction of the Frame Dragging Formula due to the Precession of the Coordinate Axes near a Rotating Body

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The known total precession (geodetic precession plus frame dragging) gives the value with respect to the chosen coordinate axes, while the precession of the coordinate axes with respect to the distant stars is disregarded. We find the precession of the coordinate axes with respect to the distant stars and combined with the known precession with respect to the coordinate axes gives the required precession with respect to the distant stars, which is a Lorentz invariant result. This yields the known value for the geodetic precession and 75% of the known value for the frame dragging effect (30 mas/year for Gravity Probe B).

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I. THE MAIN SETTINGS

(a) The basic knowledge and designations:

Let us denote by \mathbf{S} the spin vector of a gyroscope. It is known that it is Fermi-Walker transported along the world's line. In three dimensions it yields

$$d\mathbf{S}/d\tau = \boldsymbol{\Omega} \times \mathbf{S}, \quad (1)$$

where

$$\boldsymbol{\Omega} = -\frac{1}{2} \frac{\mathbf{v} \times \mathbf{a}}{c^2} + (\gamma + \frac{1}{2}) \frac{\mathbf{v} \times \nabla U}{c^2} - \frac{\gamma + 1 + \frac{\alpha_1}{4}}{4} c \nabla \times \mathbf{g}, \quad (2)$$

and $\mathbf{g} = g_{0i} \mathbf{e}_i$. As in [1] we neglect the symmetric part of the total precession since it is much smaller and periodic for circular orbits. So, our main concern is the antisymmetric part.

The first term on the right side in (2) denotes the Thomas precession, which disappears for free-fall orbits. The second term is the geodetic precession. Notice that the formula for the geodetic precession was measured to about 0.7% using Lunar laser ranging data [2, 3, 4] by considering the Earth-Moon system as a gyroscope with its axis perpendicular to the orbital plane. With the same precision it was recently also confirmed by the Gravity Probe B experiment.

The third term in (2) is the effect of frame dragging of inertia. We can replace there $\gamma = 1$ and $\alpha_1 = 0$ according to the General Relativity. Then [5]

$$\boldsymbol{\Omega} = 2\nabla \times \mathbf{V}, \quad (3)$$

where

$$V_i = \frac{G}{c^2} \int \frac{\rho' v'_i}{|\mathbf{x} - \mathbf{x}'|} d^3 x', \quad i = 1, 2, 3. \quad (4)$$

By solving the corresponding integral, this yields

$$\boldsymbol{\Omega} = -\frac{G}{c^2 r^3} [\mathbf{M} - 3\hat{\mathbf{n}}(\mathbf{M} \cdot \hat{\mathbf{n}})], \quad (5)$$

where \mathbf{M} is the angular momentum of the gravitational spinning body (the Earth), and $\hat{\mathbf{n}}$ is unit radial vector of the gyroscope. In the case of Gravity Probe B [6], i.e. for polar motion with constant velocity, this vector averages to

$$\langle \boldsymbol{\Omega} \rangle = \frac{G}{2c^2 r^3} \mathbf{M}. \quad (6)$$

(b) Statements that we work with as either proven or acceptable truth:

1. We accept the well known formula for the geodetic precession. This formula is also experimentally confirmed independently in case of LLR and also in case of Gravity Probe B.

2. If we choose a special coordinate system, where the gravitational body with zero angular momentum rests, then the frame dragging effect disappears with respect to this special coordinate system.

3. The total spin precession of any gyroscope with respect to the far stars is observed to be the same for any observer who does not move with velocities close to c and does not significantly change the gravitational potential of the celestial bodies. In other words, if the rotation is given by the 3-vector \mathbf{w} , then the magnitude of this vector may differ from one coordinate system to another by a value of order w/c^2 . We will prove in c) this statement.

4. We assume a weak gravitational field, where the principle of superposition may be applied.

(c) Proof of the statement in 3:

Let the observers be in different non-rotating inertial systems. Assume that the gyroscope moves with 4-vector of velocity V_i . Let us consider an orthonormal tetrad $A_{i\alpha}$ of 4 vectors, $A_{i\alpha}$ is the i -th component of the α -th vector. The generality is not lost if we set $A_{i0} = V_i$. If the 3-vector of velocity is $\mathbf{v} = (0, 0, 0)$, then in the chosen coordinate system the corresponding 3-vector $\frac{1}{2}(A'_{32} - A'_{23}, A'_{13} - A'_{31}, A'_{21} - A'_{12})$ represents angular velocity, where prim denotes differentiation by the time t . More generally, if \mathbf{v} is an arbitrary vector of magnitude $v \ll c$, then the same vector, neglecting the terms of order $\frac{1}{c^2} A'_{ij}$, represents a vector along the axis of rotation in 3-dimensions.

Assume that there is a Lorentz transformation of velocity u in the x -axis given by the matrix L , and let us

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denote $\bar{A}_{i\alpha} = L_{ij}A_{j\alpha}$. Notice that neglecting the terms of order w/c^2 the derivation by the time coordinate remains the same in different coordinate systems. Further

$$\begin{aligned} & \frac{1}{2}(\bar{A}'_{32} - \bar{A}'_{23}, \bar{A}'_{13} - \bar{A}'_{31}, \bar{A}'_{21} - \bar{A}'_{12}) = \\ & \frac{1}{2}(L_{3i}A'_{i2} - L_{2i}A'_{i3}, L_{1i}A'_{i3} - L_{3i}A'_{i1}, L_{2i}A'_{i1} - L_{1i}A'_{i2}) = \\ & = \frac{1}{2}(A'_{32} - A'_{23}, \end{aligned}$$

$$L_{11}A'_{13} + L_{10}A'_{03} - A'_{31}, A'_{21} - L_{11}A'_{12} - L_{10}A'_{02}).$$

First we notice that neglecting the terms of order w/c^2 , $L_{11}A'_{13} = A'_{13}$ and $L_{11}A'_{12} = A'_{12}$. By differentiating the equality $A_{ij}A_{i0} = \delta_{j0} = 0$ for $j = 2$ and $j = 3$ and assuming that the velocity of the gyroscope is a constant in an inertial system, we obtain $A'_{ij}A_{i0} = 0$, $A'_{0j} = -(A'_{1j}A_{10} + A'_{2j}A_{20} + A'_{3j}A_{30})/A_{00}$. Hence $A'_{02} \sim wv/c$, $A'_{03} \sim wv/c$, and thus $L_{10}A'_{03} \sim wuv/c^2$ and $L_{10}A'_{02} \sim wuv/c^2$ and these terms can be neglected. So neglecting the terms of order w/c^2 the 3-vector $\frac{1}{2}(A'_{32} - A'_{23}, A'_{13} - A'_{31}, A'_{21} - A'_{12})$ is invariant 3-vector for both observers. Notice here that w , i.e. A'_{ij} , ($1 \leq i, j \leq 3$), may be of arbitrary magnitude, but we neglect w/c^2 .

To finish the proof, we should consider also a transformation caused by the non-flat metric, i.e.

$$\text{diag}(-1 - \lambda, 1 - \lambda, 1 - \lambda, 1 - \lambda),$$

where $\lambda \sim 1/c^2$. But, it is easy to see that in this case we come to the same conclusion as previously.

II. THE CORRECTED FRAME DRAGGING VALUE

We will consider the Lorentz invariance of both the effects (geodetic precession and the frame dragging) simultaneously. The frame dragging phenomena is considered to be analogous to the magnetic field, which is not Lorentz invariant alone. Moreover, this effect also influences to the change of the trajectory of motion of the other bodies by the acceleration [7]

$$\mathbf{a}_i = (2 + 2\gamma) \sum_j \frac{Gm_j}{c^2 r_{ij}^3} \mathbf{v}_i \times (\mathbf{v}_j \times \mathbf{r}_{ij}), \quad (7)$$

where \mathbf{v}_i and \mathbf{v}_j are the velocities of the i -th and j -th body respectively in the chosen asymptotic inertial coordinate system, \mathbf{r}_{ij} is the radius vector from the i -th body to the j -th body, m_j is the mass of the j -th body, and \mathbf{a}_i is the acceleration of the i -th body. Thus the frame dragging effect is also called gravitomagnetic effect.

Now let us consider the geodetic precession and the frame dragging effect in the case of Gravity Probe B. We

choose a coordinate system such that the barycentre of the Earth rests according to it. Imagine the Earth as a large number of particles (atoms). Let m_i be the mass of the i -th particle, r_i be the distance from its center to the gyroscope, and the 3-vector of velocity of the i -th particle be $\mathbf{u}_i = (u_{ix}, u_{iy}, u_{iz})$, where the index i refers to the i -th particle.

It is easy to see that the vector $\boldsymbol{\Omega}$ from (2), neglecting the Thomas precession, can be written in the following form

$$\begin{aligned} \boldsymbol{\Omega} &= \sum_i \boldsymbol{\Omega}_i = \frac{3}{2} \sum_i \frac{\mathbf{v} \times \mathbf{a}_i}{c^2} + \\ &+ 2 \sum_i \left(\frac{u_{i3}}{c^2} \frac{\partial \frac{Gm_i}{r_i}}{\partial y} - \frac{u_{i2}}{c^2} \frac{\partial \frac{Gm_i}{r_i}}{\partial z}, \frac{u_{i1}}{c^2} \frac{\partial \frac{Gm_i}{r_i}}{\partial z} - \frac{u_{i3}}{c^2} \frac{\partial \frac{Gm_i}{r_i}}{\partial x}, \right. \\ &\quad \left. \frac{u_{i2}}{c^2} \frac{\partial \frac{Gm_i}{r_i}}{\partial x} - \frac{u_{i1}}{c^2} \frac{\partial \frac{Gm_i}{r_i}}{\partial y} \right) = \\ &= \frac{3}{2} \sum_i \frac{\mathbf{v} \times \mathbf{a}_i}{c^2} - 2 \sum_i \frac{\mathbf{u}_i \times \mathbf{a}_i}{c^2}, \end{aligned} \quad (8)$$

where \mathbf{a}_i is the Newtonian acceleration vector of the gyroscope toward the i -th particle. According to the General Relativity the precession given by (8) is a covariant result in sense of the curved space. This covariance means that in each coordinate system the total precession is given by (8), i.e. the total precession relative to the coordinate axes in the chosen coordinate system. But this relative precession does not a priori give the precession relative to the distant stars (as IM Pegasi in case of Gravity Probe B). The required precession with respect to the distant stars must be Lorentz (almost) invariant as it was proved in c). The formula (8) would be Lorentz invariant if the coefficients 2 and 3/2 in front of the two sums of the right side were equal. So in the chosen coordinate system appears a vector field $\boldsymbol{\Omega}$ analogous to the vector (2) in the same manner as there appears a Newtonian vector field of acceleration. This vector field $\boldsymbol{\Omega}$, which corresponds to the coordinate axes (but not to the gyroscope), must be taken into account. We have a similar situation if two points A and B are moving with 3-vector of velocities \mathbf{v} and \mathbf{u} . Then the vector \mathbf{v} alone will not be a Lorentz invariant, although the corresponding 4-vector is Lorentz invariant, but the difference $\mathbf{v} - \mathbf{u}$, i.e. the relative velocity of A with respect to B , will be Lorentz invariant (ignoring the terms of order c^{-2} and smaller). The 4-vector of velocity of A gives the velocity of A with respect to the chosen coordinate system, but not with respect to the point B . To avoid the previous discrepancy with the precession of the spin axis of the gyroscope we give the following approach.

At the i -th particle we consider an observer O_i . According to the observer O_i for the gravitation influence

of the mass m_i , he/she observes only geodetic precession with velocity $\mathbf{v} - \mathbf{u}_i$, i.e. $\boldsymbol{\Omega}_i = \frac{3}{2} \frac{(\mathbf{v} - \mathbf{u}_i) \times \mathbf{a}_i}{c^2}$, because the gyroscope moves with velocity $\mathbf{v} - \mathbf{u}_i$ with respect to him/her. But he/she does not observe any frame dragging effect because the gravitation field is static with respect to his/her frame. In section 1. we proved that **the same precession of the spin axis is observed by any other observer** moving with small velocities compared to the velocity of light, in order to remain clear of special-relativistic effects and not changing significantly the neighboring gravitational potential. So, according to this, and the possibility of superposition of weak effects, the total precession of the spin axis of the gyroscope equals to

$$\begin{aligned} \boldsymbol{\Omega} &= \sum_i \boldsymbol{\Omega}_i = \frac{3}{2} \frac{(\mathbf{v} - \mathbf{u}_i) \times \mathbf{a}_i}{c^2} = \\ &= \frac{3}{2} \sum_i \frac{\mathbf{v} \times \mathbf{a}_i}{c^2} - \frac{3}{2} \sum_i \frac{\mathbf{u}_i \times \mathbf{a}_i}{c^2}. \end{aligned} \quad (9)$$

Compared with (8) this yields the same geodetic precession and 75% of the frame dragging effect. In case of the Gravity Probe B experiment this yields 30 milli arcseconds per year. The Lorentz invariance now follows from the fact that if we replace \mathbf{v} and \mathbf{u}_i by $\mathbf{v} - \mathbf{w}$ and $\mathbf{u}_i - \mathbf{w}$ respectively, the right side of (9) is invariant.

There are two ways to solve the mentioned problem. One of the solution was presented previously, where the frame dragging effect was calculated not by using only one coordinate system, but using different "convenient" coordinates systems. Although the previous calculation was done using many different observers O_i , the same conclusion can be obtained via a single observer from any coordinate system as we will present in the second solution of the problem. The mentioned problem appeared because the axes of the chosen coordinate system are also precessing locally. So we must also calculate the apparent precession of the distant stars with respect to the chosen coordinate system and then this precession should be subtracted from the precession given by (8). In that way we will obtain the precession of the gyroscope with respect to the far stars. The precession of the axes includes the correction that should be made in obtaining the real value for the effect and it applies to all celestial bodies equally, observed from the same point of the coordinate system. The precession of the gyroscope is determined from the place of the gyroscope in a short time interval. In order to find the required correction, let us imagine a gyroscope which moves with velocity $-\mathbf{u}_i$, i.e. which rests with respect to the i -th particle. Thus by putting $\mathbf{v} = -\mathbf{u}_i$ in (8) we obtain that the instantaneous angular velocity measured from the chosen coordinate system is

$$\boldsymbol{\Omega}'_i = -\frac{1}{2} \frac{\mathbf{u}_i \times \mathbf{a}_i}{c^2}. \quad (10)$$

On the other hand, since the imagined gyroscope rests with respect to the i -th particle, its total precession is

zero, if we consider only the i -th particle. Thus the angular velocity (10) is the required systematic error observed from the chosen coordinate system considering only the i -th particle. The required total precession of the distant stars is observed to be

$$\boldsymbol{\Omega}' = -\frac{1}{2} \sum_i \frac{\mathbf{u}_i \times \mathbf{a}_i}{c^2}. \quad (11)$$

Now the difference $\boldsymbol{\Omega} - \boldsymbol{\Omega}'$ between (8) and (11) yields (9).

According to (10), the precession of the coordinate axes caused by the i -th particle, observed from any system whose axes do not precess, for example observers far from gravitation, is equal to

$$\frac{1}{2} \frac{\mathbf{u}_i \times \mathbf{a}_i}{c^2}. \quad (12)$$

This can be confirmed also by the observer O_i at the i -th particle. Namely, the frame of the unit tangent vectors of the coordinate axes may be considered as a gyroscope and since it is not freely-falling, the Thomas precession will be observed. Following [8], we can replace \mathbf{v} by $-\mathbf{u}_i$ (the velocity of the local system with respect to the observer O_i) and \mathbf{a} by \mathbf{a}_i in the first term from the right side of (2), and then we obtain the angular velocity (12).

Notice that the observed precession (12) of the coordinate axes, i.e. the frame of the unit tangent vectors of the coordinate axes at a point P is Lorentz invariant effect, because the Newton acceleration \mathbf{a}_i at the point P toward the i -th particle is Lorentz invariant and \mathbf{u}_i , which is a relative velocity of the i -th particle with respect to the point P is also Lorentz invariant.

Finally we give the following discussion and explain why this consideration is not in a collision with the influence of the acceleration (7) in change of the distance Earth-Moon, which is very precisely confirmed [7] via LLR. Basically, the made correction is due to the precession of the gyroscope, and it does not change the formula (7) for the acceleration.

It is much better to consider the acceleration (7) as analogous to the Coriolis force than to compare it with a magnetic field. The acceleration (7), as well as the Coriolis force, appears as a part of global acceleration which yields Lorentz invariant equations of motion in different coordinate systems. Now let us imagine the Gravity Probe B experiment in a rotating system, without gravitation. Then the free-fall gyroscope will not change its axis with respect to the distant stars because all the forces in the rotating system are inertial, although the observer from the rotating system observes precession of the gyroscope with respect to the coordinate axes. This example confirms the statement that from each coordinate system the observed and measured precession is related to the coordinate axes, but not to the distant stars. The inertial Coriolis force has influence to the motion of the gyroscope with respect to the rotating system. Hence the conclusion that the Coriolis force does not influence

to the twist of the spin axis. We have a similar situation with the acceleration (7) and the twist of the spin axis according to the frame dragging effect. The acceleration (7) in the chosen coordinate system with its origin at the barycentre of the three bodies (the Sun, the Earth and the Moon) changes the distance Earth-Moon about 6 meters [7], but this can not be a verification of the existence of the frame dragging effect [9], because the chosen coordinate system is not in a free-fall motion. Indeed, in another coordinate system the gravitomagnetic effect can disappear. While the acceleration (7) can disappear in one coordinate system, the total precession of the spin axis with respect to the far stars caused by both relativistic effects, must be (almost) the same in all coordinate systems.

III. CONCLUSION

We draw attention to a problem while deriving the precession for the frame dragging effect. Namely, the precession is expressed by the spin vector \mathbf{S} only with

respect to the chosen coordinate axes. The precession of the coordinate axes with respect to the distant stars is disregarded which should not be the case. We resolve the problem in two different ways. We introduce a method of many coordinate systems attached to every particle of the Earth. The participation of each particle of the Earth in the total precession (the geodetic precession plus the frame dragging) is considered while observing from the corresponding particle. So we obtained the same value for the geodetic precession as predicted by the General Relativity and 75% of the general-relativistic value for the frame dragging effect. In the case of the Gravity Probe B experiment this yields 30 milli arcseconds per year. The same result is obtained also by finding the precession of the coordinate axes and its subtraction from the known formula for the total precession.

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